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Maximum ~~like~~likelihood estimation of ordered probabilities

II

by

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4. Introduction

The problem treated in this report concerns the maximum likelihood estimation of partially or completely ordered probabilities.

Consider k independent series of independent trials, each trial resulting in a success or a failure. The i -th series consists of n_i trials with a_i ¹⁾ successes and $b_i = n_i - a_i$ failures;

π_i is the (unknown) probability of a success for each trial of the i -th series ($i = 1, 2, \dots, k$) and $\pi_1, \pi_2, \dots, \pi_k$ satisfy the inequalities

$$(1.1) \quad \alpha_{i,j} (\pi_i - \pi_j) \leq 0 \quad (i, j = 1, 2, \dots, k),$$

where

$$(1.2) \quad \begin{cases} 1. \alpha_{i,j} = -\alpha_{j,i}, \\ 2. \alpha_{i,j} = 0 \quad \text{for } m_0 \text{ pairs of values } (i, j) \text{ with } i < j, \\ 3. \alpha_{i,j} = 1 \quad \text{for } m_1 \text{ pairs of values } (i, j) \text{ with } i < j, \end{cases}$$

$$(1.3) \quad \begin{cases} m_0 + m_1 = \binom{k}{2}, \\ m_1 \geq 1 \end{cases}$$

and, if $i < h < j$ then

$$(1.4) \quad \alpha_{i,j} = 1 \quad \text{if} \quad \alpha_{i,h} = \alpha_{h,j} = 1. \quad (\text{transitivity})$$

In section 2 and 3 methods will be described by means of which the maximum likelihood estimates of $\pi_1, \pi_2, \dots, \pi_k$ may be found, i.e. the values of $x_1, x_2, \dots, x_k$ which maximize

$$(1.5) \quad L = L(x_1, x_2, \dots, x_k) \stackrel{\text{def}}{=} \sum_{i=1}^k \{a_i \lg x_i + b_i \lg (1-x_i)\}$$

in the domain

$$(1.6) \quad D : \quad \begin{cases} \alpha_{i,j} (x_i - x_j) \leq 0 \\ 0 \leq x_i \leq 1 \end{cases} \quad (i, j = 1, 2, \dots, k).$$

Unless explicitly stated otherwise L will only be considered in this domain D ; the maximum likelihood estimates will be denoted by $p_1, p_2, \dots, p_k$,

$$(1.7) \quad L_i = L_i(x_i) \stackrel{\text{def}}{=} a_i \lg x_i + b_i \lg (1-x_i) \quad (i = 1, 2, \dots, k)$$

and

$$(1.8) \quad f_i \stackrel{\text{def}}{=} \frac{a_i}{n_i} \quad (i = 1, 2, \dots, k).$$

1) Random variables will be distinguished from numbers (e.g. from the value they take in an experiment) by underlining their symbols.

Further the restrictions $\pi_i \leq \pi_j$ satisfying

$$(1.9) \quad \alpha_{i,h} \alpha_{h,j} = 0 \quad \text{for each } h \text{ between } i \text{ and } j$$

will be denoted by $R_1, R_2, \dots, R_s$. Because of the transitivity relations (1.4) the system $R_1, R_2, \dots, R_s$ is equivalent to (1.1).

In section 4 some examples will be given.

Remarks:

1. It will be clear that ordered probabilities can always be numbered in such a way that they satisfy (1.1).
2. Every set of restrictions $R_1, R_2, \dots, R_s$ represents a convex domain.
3. If $m_0 = 0$ then (1.1) is equivalent to

$$(1.10) \quad \pi_1 \leq \pi_2 \leq \dots \leq \pi_k.$$

This case has been solved independently by MIRIAM AYER, H.D. BRUNK, G.M. EWING, W.T. REID, EDWARD SILVERMAN [2] and the present author [1].

2. The maximum likelihood estimates of $\pi_1, \pi_2, \dots, \pi_k$

In this section the following theorem will be proved.

Theorem I: L possesses a unique maximum; if R_λ implies: $\pi_{i_\lambda} \leq \pi_{j_\lambda}$ and if $p'_1, p'_2, \dots, p'_k$ are the maximum likelihood estimates of $\pi_1, \pi_2, \dots, \pi_k$ under the restrictions $R_1, \dots, R_{\lambda-1}, R_{\lambda+1}, \dots, R_s$ then

$$(2.1) \quad \begin{cases} 1. p_i = p'_i & (i = 1, 2, \dots, k) \text{ if } p'_{i_\lambda} \leq p'_{j_\lambda}, \\ 2. p_{i_\lambda} = p_{j_\lambda} & \text{if } p'_{i_\lambda} > p'_{j_\lambda}. \end{cases}$$

Proof: The uniqueness of the maximum of L will be proved by induction. If $s = 0$ then it is well known that L possesses for any k a unique maximum and

$$(2.2) \quad p_i = f_i \quad (i = 1, 2, \dots, k).$$

Now suppose that L possesses a unique maximum for the following two cases

$$(2.3) \quad \begin{cases} 1. k \text{ series of trials with } s-1 \text{ restrictions} \\ 2. k-1 \text{ series of trials with } s-1 \text{ or less restrictions} \end{cases}$$

where $k \geq 2, s \geq 1$ and consider a case with k series of trials and s restrictions. Then it follows from (2.3.1) that L possesses a unique maximum under the restrictions $R_1, \dots, R_{\lambda-1}, R_{\lambda+1}, \dots, R_s$, i.e. there exists exactly one set $p'_1, p'_2, \dots, p'_k$ satisfying these restrictions and maximizing L .

Now the following two cases may be distinguished

1. $p'_{i_1} \leq p'_{j_1}$; then $p'_1, p'_2, \dots, p'_k$ satisfy the restrictions $R_1, R_2, \dots, R_s$. Therefore in this case L possesses a unique maximum under the restrictions $R_1, R_2, \dots, R_s$ and

$$(2.4) \quad p_i = p'_i \quad (i = 1, 2, \dots, k).$$

2. $p'_{i_1} > p'_{j_1}$; then (2.1.2) may be proved as follows. If $x_1, x_2, \dots, x_k$ is any set satisfying the restrictions $R_1, \dots, R_{i_1-1}, R_{i_1+1}, \dots, R_s$ but different from $p'_1, p'_2, \dots, p'_k$, then

$$(2.5) \quad L(p'_1, p'_2, \dots, p'_k) > L(x_1, x_2, \dots, x_k).$$

Now consider a fixed set $x_1, x_2, \dots, x_k$ with $x_{i_1} < x_{j_1}$ and satisfying the restrictions $R_1, R_2, \dots, R_s$. Then if

$$(2.6) \quad \begin{cases} X_i(\beta) \stackrel{\text{def}}{=} (1-\beta)x_i + \beta p'_i \\ 0 \leq X_i(\beta) \leq 1 \end{cases} \quad (i = 1, 2, \dots, k)$$

we have

$$(2.7) \quad X_i(0) = x_i, \quad X_i(1) = p'_i \quad (i = 1, 2, \dots, k)$$

and for each β with $0 \leq \beta \leq 1$, $X_1(\beta), X_2(\beta), \dots, X_k(\beta)$ is a set satisfying the restrictions $R_1, \dots, R_{i_1-1}, R_{i_1+1}, \dots, R_s$ (cf. remark 2 in the foregoing section). Therefore if

$$(2.8) \quad \beta_0 \stackrel{\text{def}}{=} \frac{x_{j_1} - x_{i_1}}{x_{j_1} - x_{i_1} + p'_{i_1} - p'_{j_1}}$$

then

$$(2.9) \quad \begin{cases} 1. & 0 < \beta_0 < 1, \\ 2. & X_{i_1}(\beta_0) = X_{j_1}(\beta_0), \end{cases}$$

i.e. $X_1(\beta_0), X_2(\beta_0), \dots, X_k(\beta_0)$ is a set satisfying the restrictions $R_1, R_2, \dots, R_s$.

Further $L\{X_1(\beta), X_2(\beta), \dots, X_k(\beta)\}$ is for fixed values of $x_1, x_2, \dots, x_k$ a function of β , say $q(\beta)$, and

$$(2.10) \quad \frac{d^2 q(\beta)}{d\beta^2} = \sum_{i=1}^k (p'_i - x_i)^2 \frac{-n_i X_i(\beta)^2 + 2a_i X_i(\beta) - a_i}{X_i(\beta)^2 \{1 - X_i(\beta)\}^2} < 0.$$

From (2.5), (2.7) and (2.10) it follows then that $q(\beta)$ is in the interval $0 \leq \beta \leq 1$ an increasing function of β , i.e.

$$(2.11) \quad L\{X_1(\beta_0), X_2(\beta_0), \dots, X_k(\beta_0)\} > L(x_1, x_2, \dots, x_k).$$

Thus for each set $x_1, x_2, \dots, x_k$ with $x_{i_1} < x_{j_1}$ and satisfying the restrictions $R_1, R_2, \dots, R_s$ a set $X_1, X_2, \dots, X_k$ exists satisfying

the restrictions $R_1, R_2, \dots, R_6$ and

$$(2.12) \quad \begin{cases} 1. X_{i_\lambda} = X_{j_\lambda}, \\ 2. L(X_1, X_2, \dots, X_k) > L(x_1, x_2, \dots, x_k), \end{cases}$$

i.e. $L(x_1, x_2, \dots, x_k)$ attains its maximum under the restrictions $R_1, R_2, \dots, R_6$ for $x_{i_\lambda} = x_{j_\lambda}$. Substituting this in (1.5) the two terms with $i = i_\lambda$ and $i = j_\lambda$ reduce to one term of the form

$$(a_{i_\lambda} + a_{j_\lambda}) \lg x_{i_\lambda} + (b_{i_\lambda} + b_{j_\lambda}) \lg (1 - x_{i_\lambda}).$$

The uniqueness of the maximum of L under these restrictions then follows from (2.3.2).

By repeatedly applying theorem I and using the well known solution of the problem for the case that $S=0$ the problem may be solved. This may, however, lead to a rather complicated procedure, which in many cases may be simplified by applying the special theorems mentioned in the following section.

3. Some special theorems

Theorem II: If $\alpha_{i,j} (f_i - f_j) \leq 0$ for each pair of values (i,j) then

$$(3.1) \quad p_i = f_i \quad (i = 1, 2, \dots, k).$$

Proof: This follows immediately from the fact that in this case the maximum of L in D coincides with the maximum of L in the domain: $0 \leq x_i \leq 1$ ($i = 1, 2, \dots, k$). The theorem also follows from theorem I.

The following theorem will be immediately clear.

Theorem III: If $i_1, i_2, \dots, i_\nu$ is a set of values satisfying

$$(3.2) \quad \alpha_{i_1, i_2} = \alpha_{i_1, i_3} = \dots = \alpha_{i_1, i_\nu} = 0 \quad \text{for each } i \neq i_1, i_2, \dots, i_\nu$$

then the maximum likelihood estimates of $\pi_{i_1}, \pi_{i_2}, \dots, \pi_{i_\nu}$ are those values of $x_{i_1}, x_{i_2}, \dots, x_{i_\nu}$ which maximize $L_{i_1} + L_{i_2} + \dots + L_{i_\nu}$ in the domain

$$(3.3) \quad D: \quad \begin{cases} \alpha_{i_h, i_{h'}} (x_{i_h} - x_{i_{h'}}) \leq 0, \\ 0 \leq x_{i_h} \leq 1 \end{cases} \quad (h, h' = 1, 2, \dots, \nu).$$

For the proof of the theorems IV and V we need the following lemma

Lemma I: If $x_1, x_2, \dots, x_k$ is any set in D with

$$(3.4) \quad \begin{cases} 1. x_i < x_j, \\ 2. f_i \geq f_j, \\ 3. \alpha_{h,i} \geq \alpha_{h,j} \\ 4. \alpha_{i,h} \leq \alpha_{j,h} \end{cases} \quad \begin{array}{l} \text{for each } h < j, \\ \text{for each } h > i. \end{array}$$

for any given pair of values (i, j) then a number x exists satisfying

$$(3.5) \quad \begin{cases} 1. x_1, x_2, \dots, x_k \text{ is also a set in } \mathbb{D} \text{ if } x \text{ is substituted} \\ \text{for } x_i \text{ and } x_j, \\ 2. L_i(x) + L_j(x) > L_i(x_i) + L_j(x_j). \end{cases}$$

Proof:

The following cases may be distinguished

$$(3.6) \quad x_i < x_j \leq f_i \quad ; \text{ then take } x = x_j.$$

$$(3.7) \quad f_i \leq x_i < x_j \quad ; \text{ then take } x = x_i.$$

$$(3.8) \quad x_i < f_i < x_j \quad ; \text{ then take } x = f_i.$$

It may easily be proved that this number x satisfies (3.5.2). For (3.6) e.g. we have

$$(3.9) \quad L_j(x) = L_j(x_j)$$

and

$$(3.10) \quad x_i < x \leq f_i.$$

From (3.10) follows

$$(3.11) \quad L_i(x) > L_i(x_i)$$

and (3.5.2) follows from (3.9) and (3.11).

For the cases (3.7) and (3.8) it may be proved in an analogous way by means of (3.4.2) that x satisfies (3.5.2).

In order to prove that this number x satisfies (3.5.1) it is sufficient to prove that

$$(3.12) \quad \begin{cases} 1. \alpha_{h,i}(x_h - x) \leq 0 \text{ for each } h < i, \\ 2. \alpha_{i,h}(x - x_h) \leq 0 \text{ for each } h > i, \\ 3. \alpha_{h,j}(x_h - x) \leq 0 \text{ for each } h < j, \\ 4. \alpha_{j,h}(x - x_h) \leq 0 \text{ for each } h > j. \end{cases}$$

From the fact that x satisfies

$$(3.13) \quad x_i \leq x \leq x_j$$

and the fact that $x_1, x_2, \dots, x_k$ is a set in $\mathbb{D}$ it follows that

$$(3.14) \quad \begin{cases} 1. \alpha_{h,i}(x_h - x) \leq \alpha_{h,i}(x_h - x_i) \leq 0 \text{ for each } h < i, \\ 2. \alpha_{j,h}(x - x_h) \leq \alpha_{j,h}(x_j - x_h) \leq 0 \text{ for each } h > j. \end{cases}$$

Further it follows from (3.4.3) and (3.4.4) that

$$(3.15) \begin{cases} 1. \alpha_{i,h}(x-x_h) = \alpha_{j,h}(x-x_h) & \text{for each } h > i \text{ with } \alpha_{i,h}=1, \\ 2. \alpha_{h,j}(x_h-x) = \alpha_{h,i}(x_h-x) & \text{for each } h < j \text{ with } \alpha_{h,j}=1 \end{cases}$$

and (3.12) follows from (3.14) and (3.15).

Theorem IV: If for any pair of values (i,j) with $i < j$

$$(3.16) \quad \alpha_{i,j}(f_i - f_j) > 0$$

and

$$(3.17) \begin{cases} 1. \alpha_{i,h} = \alpha_{h,j} = 0 & \text{for each } h \text{ between } i \text{ and } j, \\ 2. \alpha_{h,i} = \alpha_{h,j} & \text{for each } h < i, \\ 3. \alpha_{i,h} = \alpha_{j,h} & \text{for each } h > j, \end{cases}$$

then

$$(3.18) \quad p_i = p_j.$$

Proof: Suppose $x_1, x_2, \dots, x_k$ is a set in $\mathbb{D}$ with

$$(3.19) \quad x_i < x_j.$$

Further we have (cf. (3.16) and (3.17))

$$(3.20) \begin{cases} 1. f_i > f_j, \\ 2. \alpha_{h,i} = \alpha_{h,j} & \text{for each } h < j, \\ 3. \alpha_{i,h} = \alpha_{j,h} & \text{for each } h > i. \end{cases}$$

From lemma I and (3.20) it follows then that a number x exists such that $x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_{j-1}, x, x_{j+1}, \dots, x_k$ is a set in $\mathbb{D}$ and

$$(3.21) \quad L_i(x) + L_j(x) > L_i(x_i) + L_j(x_j).$$

Thus for each set $x_1, x_2, \dots, x_k$ in $\mathbb{D}$ with $x_i < x_j$ a set $x'_1, x'_2, \dots, x'_k$ in $\mathbb{D}$ exists with

$$(3.22) \begin{cases} 1. x'_i = x'_j, \\ 2. L(x'_1, x'_2, \dots, x'_k) > L(x_1, x_2, \dots, x_k), \end{cases}$$

i.e. L attains its maximum for $x_i = x_j$ and (3.18) then follows from the uniqueness of this maximum.

Remarks:

4. This theorem also follows from theorem I. If R_λ represents the restriction $\pi_i \leq \pi_j$ it follows from (3.22) that L attains its maximum under the restrictions $R_1, \dots, R_{\lambda-1}, R_{\lambda+1}, \dots, R_k$ for $x_i \geq x_j$, giving $p'_i \geq p'_j$; from (2.1) then follows: $p_i = p_j$.

5. If $m_0 = 0$, i.e. if the probabilities $\pi_1, \pi_2, \dots, \pi_k$ satisfy the inequalities

$$(3.23) \quad \pi_1 \leq \pi_2 \leq \dots \leq \pi_k,$$

then each pair of values (i, j) with $j = i + 1$ satisfies (3.17). Therefore in this case we have

$$(3.24) \quad p_i = p_{i+1} \quad \text{for each } i \text{ with } f_i > f_{i+1}.$$

From theorem IV it follows that if there is a pair of values (i, j) satisfying (3.16) and (3.17) then the problem may be reduced to the case of $k-1$ series of trials with $s-1$ (or less) restrictions by substituting $x_i = x_j$ in $L(x_1, x_2, \dots, x_k)$ i.e. by pooling the i -th and j -th series of trials.

Theorem V: If (i, j) is a pair of values satisfying

$$(3.25) \quad f_i \leq f_j$$

and

$$(3.26) \quad \begin{cases} 1. \alpha_{i,j} = 0, \\ 2. \alpha_{h,i} \leq \alpha_{h,j} \\ 3. \alpha_{i,h} \geq \alpha_{j,h} \end{cases} \quad \begin{array}{l} \text{for each } h < i, \\ \text{for each } h > j, \end{array}$$

then

$$(3.27) \quad p_i \leq p_j.$$

Proof: Suppose $x_1, x_2, \dots, x_k$ is a set in $\mathbb{D}$ with

$$(3.28) \quad x_i > x_j.$$

Further we have (cf. (3.25) and (3.26));

$$(3.29) \quad \begin{cases} 1. f_i \leq f_j, \\ 2. \alpha_{h,i} \leq \alpha_{h,j} \\ 3. \alpha_{i,h} \geq \alpha_{j,h} \end{cases} \quad \begin{array}{l} \text{for each } h < i, \\ \text{for each } h > j. \end{array}$$

From lemma I it follows then in the same way as in theorem IV that for each set $x_1, x_2, \dots, x_k$ in $\mathbb{D}$ with $x_i > x_j$ a set $x'_1, x'_2, \dots, x'_k$ in $\mathbb{D}$ exists with

$$(3.30) \quad \begin{cases} 1. x'_i = x'_j, \\ 2. L(x'_1, x'_2, \dots, x'_k) > L(x_1, x_2, \dots, x_k), \end{cases}$$

i.e. L attains its maximum for $x_i \leq x_j$; (3.27) then follows from the uniqueness of the maximum.

By means of theorem V a new restriction may be introduced. This is sometimes useful as may be seen from example 2 of section 4. In the following section some examples will be given. In these examples the theorem II-V will be applied if possible and theorem I will only be used where the theorems II-V cannot be applied.

4. Examples

Example 1:

Suppose $k = 4, m_0 = 0$ ($\pi_1 \leq \pi_2 \leq \pi_3 \leq \pi_4$) and

$$(4.1) \quad \begin{cases} i & 1 & 2 & 3 & 4 \\ a_i & 4 & 3 & 10 & 8 \\ n_i & 10 & 5 & 30 & 15 \\ f_i & 0,4 & 0,6 & 0,33 & 0,53 \end{cases}.$$

From (4.1) and (3.24) it follows that

$$(4.2) \quad p_2 = p_3$$

and the problem is reduced to the case of $k-1 = 3$ series of trials by pooling the second and third series of trials:

$$(4.3) \quad \begin{cases} i & 1 & 2(+3) & 4 \\ a_i & 4 & 13 & 8 \\ n_i & 10 & 35 & 15 \\ f_i & 0,4 & 0,37 & 0,53 \end{cases}.$$

From (4.3) and (3.24) it then follows that

$$(4.4) \quad p_1 = p_2$$

and the problem is reduced to the case of $k-2 = 2$ series of trials with

$$(4.5) \quad \begin{cases} i & 1(+2+3) & 4 \\ a_i & 17 & 8 \\ n_i & 45 & 15 \\ f_i & 0,38 & 0,53 \end{cases}.$$

From (4.5), (4.2), (4.4) and theorem I then follows

$$(4.6) \quad p_1 = p_2 = p_3 = 0,38, \quad p_4 = 0,53.$$

Example 2:

Suppose $k = 5, m_1 = 6, m_0 = 4$

$$(4.7) \quad \begin{cases} i & 1 & 2 & 3 & 4 & 5 \\ a_i & 7 & 13 & 15 & 2 & 12 \\ n_i & 10 & 20 & 30 & 5 & 15 \\ f_i & 0,7 & 0,65 & 0,5 & 0,4 & 0,8 \end{cases}$$

and

$$(4.8) \quad \alpha_{1,2} = \alpha_{1,3} = \alpha_{2,4} = \alpha_{3,5} = 1.$$

Then the pair of values $i = 2, j = 4$ satisfies (3.16) and (3.17). Therefore we have

$$(4.9) \quad p_2 = p_4$$

and the problem is reduced to the case of $k-1=4$ series of trials with $m'_1 = 4, m'_2 = 2$.

$$(4.10) \quad \begin{cases} i & 1 & 2(+4) & 3 & 5 \\ a_i & 7 & 15 & 15 & 12 \\ m_i & 10 & 25 & 30 & 15 \\ f_i & 0,7 & 0,6 & 0,5 & 0,8 \end{cases}$$

and

$$(4.11) \quad \alpha_{1,2} = \alpha_{1,3} = \alpha_{2,5} = 1.$$

For these 4 series of trials the pair $i=3, j=2$ and the pair $i=2, j=5$ satisfy (3.25) and (3.26). From theorem V then follows that L attains its maximum for

$$(4.12) \quad x_1 \leq x_3 \leq x_2 \leq x_5$$

and from (4.9), (4.10) and (4.12) follows

$$(4.13) \quad p_1 = p_3 = 0,55, \quad p_2 = p_4 = 0,6, \quad p_5 = 0,8.$$

Example 3:

Suppose $k=4, m_0 = m_1 = 3$,

$$(4.14) \quad \begin{cases} i & 1 & 2 & 3 & 4 \\ a_i & 7 & 18 & 13 & 10 \\ m_i & 10 & 30 & 20 & 25 \\ f_i & 0,7 & 0,6 & 0,65 & 0,4 \end{cases}$$

and

$$(4.15) \quad \alpha_{1,2} = \alpha_{1,4} = \alpha_{3,4} = 1.$$

For this case the theorems II-V cannot be applied and therefore we use theorem I.

Take $i_1 = 1$ and $j_1 = 4$ (i.e. omit the restriction $\pi_1 \leq \pi_4$), then p'_1, p'_2, p'_3, p'_4 are those values of x_1, x_2, x_3, x_4 which maximize L in the domain

$$(4.16) \quad \begin{cases} x_1 \leq x_2, \quad x_3 \leq x_4 \\ 0 \leq x_i \leq 1 \quad (i=1,2,3,4). \end{cases}$$

From theorem III and IV then follows

$$(4.17) \quad p'_1 = p'_2 = 0,63, \quad p'_3 = p'_4 = 0,51$$

and from theorem I and (4.17) (cf. (2.1.2))

$$(4.18) \quad p_1 = p_4.$$

In this way the problem is reduced to the case of $k-1=3$ series of trials with

$$(4.19) \quad \begin{cases} i & 3 & 1(+4) & 2 \\ a_i & 13 & 17 & 18 \\ n_i & 20 & 35 & 30 \\ f_i & 0,65 & 0,49 & 0,6 \end{cases}$$

and

$$(4.20) \quad \alpha_{3,1} = \alpha_{1,2} = 1.$$

From (4.18), (4.19) and (4.20) follows

$$(4.21) \quad p_1 = p_3 = p_4 = 0,55, \quad p_2 = 0,6.$$

Example 4:

Suppose $k=8, m_0=13, m_1=15$

$$(4.22) \quad \begin{cases} i & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ a_i & 8 & 22 & 13 & 25 & 20 & 21 & 32 & 2 \\ n_i & 10 & 40 & 20 & 50 & 30 & 50 & 50 & 5 \\ f_i & 0,8 & 0,55 & 0,65 & 0,5 & 0,67 & 0,42 & 0,64 & 0,4 \end{cases}$$

and

$$(4.23) \quad \alpha_{1,2} = \alpha_{1,4} = \alpha_{2,4} = \alpha_{4,5} = \alpha_{5,6} = \alpha_{5,8} = \alpha_{7,8} = 1.$$

For this case the theorems II-V cannot be applied and therefore we use theorem I.

If we take $i_1=4$ and $j_1=5$ then it follows from theorem III that $p'_1, p'_2, \dots, p'_8$ are found by maximizing $\sum_{i=1}^4 L_i$ under the restrictions: $x_1 \leq x_2, x_1 \leq x_4, x_3 \leq x_4$ and $\sum_{i=5}^8 L_i$ under the restrictions: $x_5 \leq x_6, x_5 \leq x_8, x_7 \leq x_8$.

In order to find p'_1, p'_2, p'_3, p'_4 we apply theorem I; this results in (cf. example 3)

$$(4.24) \quad p'_1 = p'_2 = p'_3 = p'_4 = 0,57.$$

In an analogous way we find by means of theorem I

$$(4.25) \quad p'_5 = p'_6 = 0,51, \quad p'_7 = p'_8 = 0,62.$$

From (4.24), (4.25) and theorem I then follows (cf. (2.1.2))

$$(4.26) \quad p_4 = p_5$$

and the problem is reduced to the case of $k-1=7$ series of trials with $m'_0=10, m'_1=11$

$$(4.27) \quad \begin{cases} i & 1 & 2 & 3 & 4(+5) & 6 & 7 & 8 \\ a_i & 8 & 22 & 13 & 45 & 21 & 32 & 2 \\ n_i & 10 & 40 & 20 & 80 & 50 & 50 & 5 \\ f_i & 0,8 & 0,55 & 0,65 & 0,56 & 0,42 & 0,64 & 0,4 \end{cases}$$

and

$$(4.28) \quad \alpha'_{1,2} = \alpha'_{1,4} = \alpha'_{3,4} = \alpha'_{4,6} = \alpha'_{4,8} = \alpha'_{7,8} = 1.$$

For these 7 series of trials we again apply theorem I; taking $i_\lambda = 1$ and $j_\lambda = 4$ one finds by means of the theorems II-V

$$(4.29) \quad p'_1 = p'_2 = 0,6, \quad p'_3 = p'_4 = p'_6 = 0,53, \quad p'_7 = p'_8 = 0,62.$$

From (4.29) and theorem I then follows

$$(4.30) \quad p_1 = p_4$$

and the problem is reduced to the case of $k-2=6$ series of trials with $m''_0 = 7, m''_i = 8$

$$(4.31) \quad \begin{cases} i & 3 & 1(+4+5) & 2 & 6 & 7 & 8 \\ \alpha''_i & 13 & 53 & 22 & 21 & 32 & 2 \\ n''_i & 20 & 90 & 40 & 50 & 50 & 5 \\ f''_i & 0,65 & 0,59 & 0,55 & 0,42 & 0,64 & 0,4 \end{cases}$$

and

$$(4.32) \quad \alpha''_{3,1} = \alpha''_{1,2} = \alpha''_{1,6} = \alpha''_{1,8} = \alpha''_{7,8} = 1.$$

From (4.31), (4.32) and theorem IV then follows

$$(4.33) \quad p_3 = p_1$$

and the problem is reduced to the case of $k-3=5$ series of trials with $m'''_0 = 6, m'''_i = 4$

$$(4.34) \quad \begin{cases} i & 1(+3+4+5) & 2 & 6 & 7 & 8 \\ \alpha'''_i & 66 & 22 & 21 & 32 & 2 \\ n'''_i & 110 & 40 & 50 & 50 & 5 \\ f'''_i & 0,6 & 0,55 & 0,42 & 0,64 & 0,4 \end{cases}$$

and

$$(4.35) \quad \alpha'''_{1,2} = \alpha'''_{1,6} = \alpha'''_{1,8} = \alpha'''_{7,8} = 1.$$

From theorem V it follows then that L attains its maximum for $x_1 \leq x_7$. Introducing this new restriction it follows from theorem IV that

$$(4.36) \quad p_7 = p_8$$

and the problem is reduced to the case of $k-4=4$ series of trials with $m''''_0 = m''''_1 = 3$

$$(4.37) \quad \begin{cases} i & 1(+3+4+5) & 2 & 6 & 7(+8) \\ \alpha''''_i & 66 & 22 & 21 & 34 \\ n''''_i & 110 & 50 & 40 & 55 \\ f''''_i & 0,6 & 0,55 & 0,42 & 0,62 \end{cases}$$

and

$$(4.38) \quad \alpha_{1,2}^{''''} = \alpha_{1,6}^{''''} = \alpha_{1,7}^{''''} = 1.$$

From theorem V then follows that L attains its maximum for $x_1 \leq x_6 \leq x_2 \leq x_7$; thus we have

$$(4.39) \quad p_1 = p_3 = p_4 = p_5 = p_6 = 0.54; \quad p_2 = 0.55; \quad p_7 = p_8 = 0.62.$$

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